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# Positive solutions and monotone iterative sequences for a fractional differential equation with integral boundary conditions

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Full list of author information is available at the end of the article**Abstract**

Using a monotone iterative method combined with some inequalities associated with the Green's function, we investigate the existence of positive solutions for a fractional differential equation with integral boundary conditions. In addition, two examples are given to illustrate the results.

**MSC:** 34A08; 34B10; 34B15**Keywords:** positive solution; fractional differential equations; boundary value problems; integral boundary conditions; monotone iterative method

## 1 Introduction

This paper discusses the existence and iterative method of positive solutions for the following nonlinear fractional differential equations with integral boundary condition:

$$\begin{cases} D_{0+}^{\alpha} u(t) + q(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u^{(j)}(0) = 0, & 0 \leq j \leq n-2, \quad u(1) = \mu \int_0^1 u(s) ds, \end{cases} \quad (1.1)$$

where  $\alpha \in (n-1, n]$  is a real number and  $n \geq 3$  is an integer,  $\mu$  is a parameter and  $0 \leq \mu < \alpha$ ,  $D_{0+}^{\alpha}$  is the standard Riemann-Liouville fractional derivative of order  $\alpha$ . A function  $u$  is called a positive solution of the problem (1.1) if  $u(t)$  satisfies (1.1) and  $u(t) > 0$  on  $(0, 1)$ .

Fractional differential equations arise in many engineering and scientific disciplines such as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro-dynamics of a complex medium, polymer rheology, and so on. Recently, the subject of fractional differential equations has gained much more importance and attention. Some excellent work in the study of fractional differential equations can be found in [1–22] and the references cited therein. Integral boundary conditions have various applications in chemical engineering, thermo-elasticity, population dynamics, and so on. Boundary value problems for fractional differential equations with integral boundary conditions are very interesting and largely unknown. Recently, by using Guo-Krasnoselskii's fixed point theorem, Cabada and Wang in [5] investigated the existence of positive solutions for the fractional boundary value problem

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \lambda \int_0^1 u(s) ds, \end{cases}$$

where  $2 < \alpha \leq 3$ ,  $0 < \lambda < 2$ ,  ${}^C D_{0+}^\alpha$  is the Caputo fractional derivative and  $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is a continuous function. Karakostas [10] provided sufficient conditions for the non-existence of solutions of the boundary-value problems with fractional derivative of order  $\alpha \in (2, 3)$  in the Caputo sense,

$$\begin{cases} {}^C D_{0+}^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = \lambda \int_0^1 u(s) ds. \end{cases}$$

Motivated by the works mentioned above, our purpose in this paper is to show the existence and iteration of positive solutions to the problem (1.1) by using a monotone iterative method. The method used in this paper is different from that used in [20]. We not only obtain the existence of positive solutions, but also give two iterative schemes approximating the solutions, and the iterative scheme starts off with a known simple function or the zero function, which is interesting because it gives a numerical method to compute approximate solutions. The monotone iterative method has been successfully applied to boundary-value problems of integer-order ordinary differential equations (see [23–27] and the references therein). To our knowledge, there is still little utilization of the monotone iterative method to study the existence of positive solutions for nonlinear fractional boundary-value problems. So, it is worth investigating the problem (1.1) by using monotone iterative method.

## 2 Preliminaries

Let us recall some basic definitions on fractional calculus.

**Definition 2.1** ([28, 29]) The Riemann-Liouville fractional derivative of order  $\alpha > 0$  of a continuous function  $h : [0, \infty) \rightarrow \mathbb{R}$  is defined to be

$$D_{0+}^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} h(s) ds, \quad n = [\alpha] + 1,$$

where  $\Gamma$  denotes the Euler gamma function and  $[\alpha]$  denotes the integer part of number  $\alpha$ , provided that the right side is pointwise defined on  $(0, \infty)$ .

**Definition 2.2** ([28, 29]) The Riemann-Liouville fractional integral of order  $\alpha$  is defined as

$$I_{0+}^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \quad t > 0, \alpha > 0,$$

provided the integral exists.

In [17], the author obtained the Green's function associated with the problem (1.1). More precisely, the author proved the following lemma.

**Lemma 2.1** ([17]) *Let  $h \in C[0, 1]$  be a given function, then the boundary-value problem*

$$\begin{cases} D_{0+}^\alpha u(t) + h(t) = 0, & 0 \leq t \leq 1, \\ u^{(j)}(0) = 0, & 0 \leq j \leq n-2, \quad u(1) = \mu \int_0^1 u(s) ds, \end{cases}$$

has a unique solution,

$$u(t) = \int_0^1 G(t,s)h(s) ds,$$

where

$$G(t,s) = H(t,s) + \frac{\mu t^{\alpha-1}}{(\alpha-\mu)\Gamma(\alpha)} s(1-s)^{\alpha-1}, \quad t,s \in [0,1], \quad (2.1)$$

$$H(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.2)$$

Obviously,

$$G(t,s) = \frac{1}{(\alpha-\mu)\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s) - (\alpha-\mu)(t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\mu+\mu s), & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.3)$$

and  $G(t,s)$  is continuous on the unit square  $[0,1] \times [0,1]$ .

**Lemma 2.2** ([16]) *The function  $H(t,s)$  defined by (2.2) has the following properties:*

$$t^{\alpha-1}(1-t) \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leq H(t,s) \leq \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}, \quad t,s \in [0,1]. \quad (2.4)$$

**Lemma 2.3** *The Green's function  $G(t,s)$  defined by (2.1) has the following properties:*

$$0 \leq G(t,s) \leq \frac{t^{\alpha-1}}{(\alpha-\mu)\Gamma(\alpha)} (1-s)^{\alpha-1}(\alpha-\mu+\mu s), \quad t,s \in [0,1], \quad (2.5)$$

$$p(t)g(s) \leq G(t,s) \leq g(s), \quad t,s \in [0,1], \quad (2.6)$$

where

$$g(s) = \frac{(\alpha-1)(\alpha-\mu) + \mu}{(\alpha-\mu)\Gamma(\alpha)} s(1-s)^{\alpha-1}, \quad s \in [0,1],$$

$$p(t) = \frac{\alpha t^{\alpha-1}(1-t)}{(\alpha-1)(\alpha-\mu) + \mu}, \quad t \in [0,1].$$

*Proof* It is obvious from (2.3) that the right inequality of (2.5) holds. Relation (2.4) implies that  $H(t,s) \geq 0$ . Thus by (2.1) we know that the left inequality of (2.5) is correct. Now we show that (2.6) holds. In fact, by (2.1) and (2.4), we have

$$\begin{aligned} G(t,s) &= H(t,s) + \frac{\mu t^{\alpha-1}}{(\alpha-\mu)\Gamma(\alpha)} s(1-s)^{\alpha-1} \leq \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)} + \frac{\mu}{(\alpha-\mu)\Gamma(\alpha)} s(1-s)^{\alpha-1} \\ &= \frac{(\alpha-1)(\alpha-\mu) + \mu}{(\alpha-\mu)\Gamma(\alpha)} s(1-s)^{\alpha-1} = g(s), \quad t,s \in [0,1]. \end{aligned}$$

On the other hand, by (2.1) and (2.4), we get

$$\begin{aligned} G(t, s) &= H(t, s) + \frac{\mu t^{\alpha-1}}{(\alpha - \mu)\Gamma(\alpha)} s(1-s)^{\alpha-1} \\ &\geq t^{\alpha-1}(1-t) \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mu t^{\alpha-1}(1-t)}{(\alpha - \mu)\Gamma(\alpha)} s(1-s)^{\alpha-1} \\ &= \frac{s(1-s)^{\alpha-1}}{(\alpha - \mu)\Gamma(\alpha)} \alpha t^{\alpha-1}(1-t) = g(s)p(t), \quad t, s \in [0, 1]. \end{aligned}$$

Then the proof is completed.  $\square$

### 3 Main results

Now, we consider the problem (1.1). Obviously,  $u$  is a solution of the problem (1.1) if and only if  $u$  is a solution of the following nonlinear integral equation:

$$u(t) = \int_0^1 G(t, s)q(s)f(s, u(s)) ds, \quad t \in [0, 1], u \in C[0, 1],$$

where  $G(t, s)$  is the Green's function defined by (2.3). For the forthcoming analysis, we need the following assumptions:

(H1)  $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is continuous and  $f(t, 0) \not\equiv 0$  on  $[0, 1]$ ;

(H2)  $q : (0, 1) \rightarrow [0, \infty)$  is continuous and  $0 < \int_0^1 (1-s)^{\alpha-1} q(s) ds < \infty$ .

The basic space used in this paper is a real Banach space  $\mathcal{E} = C[0, 1]$  with the norm  $\|u\|$ , where  $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ . Then, define a set  $\mathcal{K} \subset \mathcal{E}$  by

$$\mathcal{K} = \{u \in C[0, 1] : u(t) \geq 0, u(t) \geq p(t)\|u\|, t \in [0, 1]\}.$$

It is obvious that  $\mathcal{K}$  is a cone. We define the operator  $\mathcal{T} : C[0, 1] \rightarrow C[0, 1]$  by

$$(\mathcal{T}u)(t) = \int_0^1 G(t, s)q(s)f(s, u(s)) ds, \quad t \in [0, 1], u \in C[0, 1]. \quad (3.1)$$

It is clear that the existence of a positive solution for the problem (1.1) is equivalent to the existence of a nontrivial fixed point of  $\mathcal{T}$  in  $\mathcal{K}$ .

**Lemma 3.1**  $\mathcal{T}$  is a completely continuous operator and  $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{K}$ .

*Proof* Applying the Arzela-Ascoli theorem and a standard argument, we can prove that  $\mathcal{T}$  is a completely continuous operator. We conclude that  $\mathcal{T}(\mathcal{K}) \subseteq \mathcal{K}$ . In fact, for any  $u \in \mathcal{K}$ , it follows from (H1), (H2), and (2.6) that

$$(\mathcal{T}u)(t) = \int_0^1 G(t, s)q(s)f(s, u(s)) ds \leq \int_0^1 g(s)q(s)f(s, u(s)) ds, \quad t \in [0, 1],$$

which implies that

$$\|\mathcal{T}u\| \leq \int_0^1 g(s)q(s)f(s, u(s)) ds. \quad (3.2)$$

On the other hand, by (H1), (H2), and (2.6) we have

$$(\mathcal{T}u)(t) = \int_0^1 G(t,s)q(s)f(s,u(s))ds \geq p(t) \int_0^1 g(s)q(s)f(s,u(s))ds, \quad t \in [0,1],$$

which together with (3.2) implies

$$(\mathcal{T}u)(t) \geq p(t)\|\mathcal{T}u\|, \quad t \in [0,1].$$

Therefore,  $\mathcal{T}u \in \mathcal{K}$ . The proof is completed.  $\square$

For convenience, we denote

$$\Lambda = \left( \frac{1}{(\alpha - \mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\alpha - \mu + \mu s)q(s)ds \right)^{-1}. \quad (3.3)$$

By (H2) we know that  $\Lambda > 0$  is well defined.

**Theorem 3.1** *Suppose (H1) and (H2) hold. In addition, we assume that there exists  $a > 0$ , such that*

$$f(t,x) \leq f(t,y) \leq \Lambda a, \quad 0 \leq x \leq y \leq a, t \in [0,1], \quad (3.4)$$

where  $\Lambda$  is given by (3.3). Then the problem (1.1) has two positive solutions  $v^*$  and  $w^*$  satisfying  $0 < \|v^*\| \leq \|w^*\| \leq a$ . In addition, the iterative sequences  $v_{k+1} = \mathcal{T}v_k$ ,  $w_{k+1} = \mathcal{T}w_k$ ,  $k = 0, 1, 2, \dots$ , converge, in  $C$ -norm, to positive solutions  $v^*$  and  $w^*$ , respectively, where  $v_0(t) = 0$ ,  $w_0(t) = at^{\alpha-1}$ ,  $t \in [0,1]$ . Moreover,

$$v_0(t) \leq v_1(t) \leq \dots \leq v_k(t) \leq \dots \leq v^*(t) \leq w^*(t) \leq \dots \leq w_k(t) \leq \dots \leq w_1(t) \leq w_0(t), \\ t \in [0,1].$$

**Remark 3.1** The iterative schemes in Theorem 3.1 start off with the zero function and a known simple function, respectively.

*Proof* The proof will be given in several steps.

Step 1. Let  $\mathcal{K}_a = \{u \in \mathcal{K} : \|u\| \leq a\}$ , then  $\mathcal{T}(\mathcal{K}_a) \subseteq \mathcal{K}_a$ .

In fact, if  $u \in \mathcal{K}_a$ , then

$$0 \leq u(s) \leq \|u\| \leq a, \quad s \in [0,1].$$

Thus by (2.5) and (3.4), we get

$$0 \leq (\mathcal{T}u)(t) = \int_0^1 G(t,s)q(s)f(s,u(s))ds \\ \leq \frac{t^{\alpha-1}}{(\alpha - \mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\alpha - \mu + \mu s)q(s)f(s,a)ds \\ \leq \frac{\Lambda a}{(\alpha - \mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\alpha - \mu + \mu s)q(s)ds = a, \quad t \in [0,1],$$

which implies that  $\|\mathcal{T}u\| \leq a$ , thus  $\mathcal{T}(\mathcal{K}_a) \subseteq \mathcal{K}_a$ .

Step 2. The iterative sequence  $\{v_k\}$  is increasing, and there exists  $v^* \in \mathcal{K}_a$  such that  $\lim_{k \rightarrow \infty} \|v_k - v^*\| = 0$ , and  $v^*$  is a positive solution of the problem (1.1).

Obviously,  $v_0 \in \mathcal{K}_a$ . Since  $\mathcal{T} : \mathcal{K}_a \rightarrow \mathcal{K}_a$ , we have  $v_k \in \mathcal{T}(\mathcal{K}_a) \subseteq \mathcal{K}_a$ ,  $k = 1, 2, \dots$ . Since  $\mathcal{T}$  is completely continuous, we assert that  $\{v_k\}_{k=1}^\infty$  is a sequentially compact set. Since  $v_1 = \mathcal{T}v_0 = \mathcal{T}0 \in \mathcal{K}_a$ , we have

$$a \geq v_1(t) = (\mathcal{T}v_0)(t) = (\mathcal{T}0)(t) \geq 0 = v_0(t), \quad t \in [0, 1].$$

It follows from (3.4) that  $\mathcal{T}$  is nondecreasing, and then

$$v_2(t) = (\mathcal{T}v_1)(t) \geq (\mathcal{T}v_0)(t) = v_1(t), \quad t \in [0, 1].$$

Thus, by the induction, we have

$$v_{k+1}(t) \geq v_k(t), \quad t \in [0, 1], k = 0, 1, 2, \dots$$

Hence, there exists  $v^* \in \mathcal{K}_a$  such that  $\lim_{k \rightarrow \infty} \|v_k - v^*\| = 0$ . By the continuity of  $\mathcal{T}$  and equation  $v_{k+1} = \mathcal{T}v_k$ , we get  $\mathcal{T}v^* = v^*$ . Moreover, since the zero function is not a solution of the problem (1.1),  $\|v^*\| > 0$ . It follows from the definition of the cone  $\mathcal{K}$ , that we have  $v^*(t) \geq p(t)\|v^*\| > 0$ ,  $t \in (0, 1)$ , i.e.  $v^*(t)$  is a positive solution of the problem (1.1).

Step 3. The iterative sequence  $\{w_k\}$  is decreasing, and there exists  $w^* \in \mathcal{K}_a$  such that  $\lim_{k \rightarrow \infty} \|w_k - w^*\| = 0$ , and  $w^*$  is a positive solution of the problem (1.1).

Obviously,  $w_0 \in \mathcal{K}_a$ . Since  $\mathcal{T} : \mathcal{K}_a \rightarrow \mathcal{K}_a$ , we have  $w_k \in \mathcal{T}(\mathcal{K}_a) \subseteq \mathcal{K}_a$ ,  $k = 1, 2, \dots$ . Since  $\mathcal{T}$  is completely continuous, we assert that  $\{w_k\}_{k=1}^\infty$  is a sequentially compact set. Since  $w_1 = \mathcal{T}w_0 \in \mathcal{K}_a$ , by (2.5) and (3.4), we have

$$\begin{aligned} (\mathcal{T}w_0)(t) &= \int_0^1 G(t, s)q(s)f(s, w_0(s)) ds \\ &\leq \frac{t^{\alpha-1}}{(\alpha-\mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s)f(s, a) ds \\ &\leq \frac{t^{\alpha-1}\Lambda a}{(\alpha-\mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}(\alpha-\mu+\mu s)q(s) ds = at^{\alpha-1} = w_0(t), \quad t \in [0, 1]. \end{aligned}$$

Thus we obtain

$$w_1(t) \leq w_0(t), \quad t \in [0, 1],$$

which together with (3.4) implies that

$$\begin{aligned} w_2(t) &= (\mathcal{T}w_1)(t) = \int_0^1 G(t, s)q(s)f(s, w_1(s)) ds \\ &\leq \int_0^1 G(t, s)q(s)f(s, w_0(s)) ds = (\mathcal{T}w_0)(t) = w_1(t), \quad t \in [0, 1]. \end{aligned}$$

By the induction, we have

$$w_{k+1}(t) \leq w_k(t), \quad t \in [0, 1], k = 0, 1, 2, \dots$$

Hence, there exists  $w^* \in \mathcal{K}_a$  such that  $\lim_{k \rightarrow \infty} \|w_k - w^*\| = 0$ . Applying the continuity of  $\mathcal{T}$  and the definition of  $\mathcal{K}$ , we can conclude that  $w^*$  is a positive solution of the problem (1.1).

Step 4. From  $v_0(t) \leq w_0(t)$ ,  $t \in [0, 1]$  we get

$$\begin{aligned} v_1(t) &= (\mathcal{T}v_0)(t) = \int_0^1 G(t,s)q(s)f(s, v_0(s)) ds \\ &\leq \int_0^1 G(t,s)q(s)f(s, w_0(s)) ds = (\mathcal{T}w_0)(t) = w_1(t), \quad t \in [0, 1]. \end{aligned}$$

By the induction, we have

$$v_k(t) \leq w_k(t), \quad t \in [0, 1], k = 0, 1, 2, \dots$$

The proof is complete.  $\square$

**Remark 3.2** Certainly,  $w^* = v^*$  may happen and then the problem (1.1) has only one solution in  $\mathcal{K}_a$ .

**Corollary 3.1** Suppose that (H1) and (H2) hold. Suppose further that  $f(t, x)$  is nondecreasing in  $x$  for each  $t \in [0, 1]$  and

$$\lim_{x \rightarrow +\infty} \max_{0 \leq t \leq 1} \frac{f(t, x)}{x} < \Lambda.$$

Then the problem (1.1) has at least two positive solutions.

## 4 Examples

To illustrate the usefulness of the results, we give two examples.

**Example 4.1** Consider the fractional boundary-value problem

$$\begin{cases} D_{0+}^{7/2} u(t) + 3u^2(t) + 6t + 1 = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, & u(1) = \frac{3}{2} \int_0^1 u(s) ds. \end{cases} \quad (4.1)$$

In this problem,

$$\alpha = 3.5, \quad \mu = 1.5, \quad q(t) \equiv 1, \quad f(t, x) = 3x^2 + 6t + 1.$$

It is easy to see that (H1) and (H2) hold. If we let  $a = 2$ , by simple computation, we have

$$\Lambda = \left( \frac{1}{(\alpha - \mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (\alpha - \mu + \mu s) q(s) ds \right)^{-1} = \frac{45\sqrt{\pi}}{8},$$

and

$$f(t, x) \leq f(t, 2) \leq f(1, 2) = 19 < \frac{45\sqrt{\pi}}{4} = \Lambda a, \quad (t, x) \in [0, 1] \times [0, a].$$

Then (3.4) is satisfied. Consequently, Theorem 3.1 guarantees that the problem (4.1) has at least two positive solutions  $v^*$  and  $w^*$ , satisfying  $0 < \|v^*\| \leq \|w^*\| \leq 2$ .

Moreover, the two iterative schemes are

$$\begin{aligned} v_0(t) &= 0, \quad t \in [0, 1], \\ v_{k+1}(t) &= \frac{2t^{5/2}}{15\sqrt{\pi}} \int_0^1 (1-s)^{5/2} (4+3s) [3v_k^2(s) + 6s + 1] ds \\ &\quad - \frac{8}{15\sqrt{\pi}} \int_0^t (t-s)^{5/2} [3v_k^2(s) + 6s + 1] ds, \quad t \in [0, 1], k = 0, 1, 2, \dots, \end{aligned}$$

and

$$\begin{aligned} w_0(t) &= 2t^{5/2}, \quad t \in [0, 1], \\ w_{k+1}(t) &= \frac{2t^{5/2}}{15\sqrt{\pi}} \int_0^1 (1-s)^{5/2} (4+3s) [3w_k^2(s) + 6s + 1] ds \\ &\quad - \frac{8}{15\sqrt{\pi}} \int_0^t (t-s)^{5/2} [3w_k^2(s) + 6s + 1] ds, \quad t \in [0, 1], k = 0, 1, 2, \dots \end{aligned}$$

After direct calculations by Matlab 7.5, the second and third terms of the two schemes are as follows:

$$\begin{aligned} v_1(t) &= \frac{4t^{5/2}}{5\sqrt{\pi}} \left( \frac{3}{11} - \frac{4}{21}t - \frac{16}{63}t^2 \right), \quad t \in [0, 1], \\ v_2(t) &= \frac{4t^{5/2}}{5\sqrt{\pi}} \left( \frac{3}{11} - \frac{4}{21}t - \frac{16}{63}t^2 \right) + \frac{64t^{15/2}}{125\pi^{3/2}} \left( \frac{3}{121} - \frac{28}{363}t - \frac{368}{14,553}t^2 + \frac{27,712}{305,613}t^3 \right. \\ &\quad \left. - \frac{2,816}{83,349}t^4 - \frac{1,024}{27,783}t^5 \right), \quad t \in [0, 1], \end{aligned}$$

and

$$\begin{aligned} w_1(t) &= \frac{t^{5/2}}{\sqrt{\pi}} \left( \frac{97,252}{415,701} - \frac{16}{105}t - \frac{64}{315}t^2 - \frac{16,384}{765,765}t^6 \right), \quad t \in [0, 1], \\ w_2(t) &= \frac{4t^{5/2}}{5\sqrt{\pi}} \left( \frac{3}{11} - \frac{4}{21}t - \frac{16}{63}t^2 \right) + \frac{64t^{15/2}}{135\pi^{3/2}} \left( \frac{591,121,969}{19,200,813,489} - \frac{20,806,384,636}{224,009,490,705}t \right. \\ &\quad - \frac{14,431,376}{509,233,725}t^2 + \frac{8,246,848}{79,214,135}t^3 - \frac{2,816}{77,175}t^4 - \frac{1,024}{25,725}t^5 - \frac{199,172,096}{35,369,919,585}t^6 \\ &\quad + \frac{1,832,321,024}{137,549,687,275}t^7 + \frac{262,144}{187,612,425}t^8 + \frac{524,288}{62,537,475}t^9 + \frac{16,777,216}{65,155,115,025}t^{12} \\ &\quad \left. - \frac{67,108,864}{152,028,601,725}t^{13} \right), \quad t \in [0, 1]. \end{aligned}$$

**Example 4.2** Consider the fractional boundary value

$$\begin{cases} D_{0+}^{9/2} u(t) + (1-t)^{-3/2} [e^{u(t)+t+1} + 2u^2(t) + 4t + 5] = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = u'''(0) = 0, & u(1) = 2 \int_0^1 u(s) ds. \end{cases} \quad (4.2)$$



In this problem,

$$\alpha = 4.5, \quad \mu = 2, \quad q(t) = (1-t)^{-3/2}, \quad f(t, x) = e^{x+t+1} + 2x^2 + 4t + 5.$$

Obviously,  $q(t)$  and  $f(t, x)$  satisfy the conditions (H1) and (H2). In addition,  $f(t, x)$  is increasing in  $x$ , and

$$\Lambda = \left( \frac{1}{(\alpha - \mu)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (\alpha - \mu + \mu s) q(s) ds \right)^{-1} = \frac{575\sqrt{\pi}}{32}.$$

Let  $a = 1$ , then for any  $(t, x) \in [0, 1] \times [0, a]$ , direct computations give

$$f(t, x) \leq f(t, 1) \leq f(1, 1) = 11 + e^3 < \frac{575\sqrt{\pi}}{32} = \Lambda a.$$

Therefore, all assumptions of Theorem 3.1 are satisfied. Thus Theorem 3.1 ensures that the problem (4.2) has two positive solutions  $v^*$  and  $w^*$ , satisfying  $0 < \|v^*\| \leq \|w^*\| \leq 1$  and  $\lim_{k \rightarrow \infty} \|v_k - v^*\| = 0$ ,  $\lim_{k \rightarrow \infty} \|w_k - w^*\| = 0$ , where

$$\begin{aligned} v_0(t) &= 0, \quad t \in [0, 1], \\ v_{k+1}(t) &= \frac{16t^{7/2}}{525\sqrt{\pi}} \int_0^1 (1-s)^2 (5+4s) [e^{v_k(s)+s+1} + 2v_k^2(s) + 4s + 5] ds \\ &\quad - \frac{16}{105\sqrt{\pi}} \int_0^t \frac{(t-s)^{7/2}}{(1-s)^{3/2}} [e^{v_k(s)+s+1} + 2v_k^2(s) + 4s + 5] ds, \\ t &\in [0, 1], k = 0, 1, 2, \dots \end{aligned}$$

and

$$\begin{aligned} w_0(t) &= t^{7/2}, \quad t \in [0, 1], \\ w_{k+1}(t) &= \frac{16t^{7/2}}{525\sqrt{\pi}} \int_0^1 (1-s)^2 (5+4s) [e^{w_k(s)+s+1} + 2w_k^2(s) + 4s + 5] ds \\ &\quad - \frac{16}{105\sqrt{\pi}} \int_0^t \frac{(t-s)^{7/2}}{(1-s)^{3/2}} [e^{w_k(s)+s+1} + 2w_k^2(s) + 4s + 5] ds, \\ t &\in [0, 1], k = 0, 1, 2, \dots \end{aligned}$$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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#### Acknowledgements

The authors sincerely thanks the anonymous referees for their valuable suggestions and comments which have greatly helped improve this article. The first author was supported financially by the Natural Science Foundation of Zhejiang Province of China (Y12A01012). The second author was supported financially second by the Foundation of Shanghai Natural Science (13ZR1430100) and the Foundation of Shanghai Municipal Education Commission (DYL201105).

Received: 23 October 2013 Accepted: 6 January 2014 Published: 27 Jan 2014

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10.1186/1687-1847-2014-29

**Cite this article as:** Sun and Sun: Positive solutions and monotone iterative sequences for a fractional differential equation with integral boundary conditions. *Advances in Difference Equations* 2014, **2014**:29